DIAGONALIZATION OF SYMMETRIC MATRIX BY JACOBY'S METHOD

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Diagonalization of symmetric matrix by Jacoby's Method

Aim: To diagonalize a given real symmetric matrix $[\boldsymbol{A_0}]$ by Jacoby's Method

What is Diagonalization?

$$[A_0] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{21} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1.96 & 0 & 0 \\ 0 & 0.57 & 0 \\ 0 & 0 & 3.317 \end{bmatrix}$$

Procedure:

1. Select the largest (in absolute value) off diagonal element of matrix $[\boldsymbol{A_0}]$ which is to be eliminated (say $\boldsymbol{a_{ij}}$)
2. Find the value of rotation angle $\boldsymbol{\theta}$ using the expression,
3. $$\boldsymbol{tan\theta} = \frac{\pm 2\, a_{ij}}{|a_{ii}-a_{jj}| + \sqrt{(a_{ii}-a_{jj})^2 + 4\, a_{ij}}}$$

Plus sign is used when $\boldsymbol{a_{ii} \geq a_{jj}}$and minus sign is used when $\boldsymbol{a_{ii} \leq a_{jj}}$. Also $\frac{\pi}{2} \geq \boldsymbol{\theta} \geq \frac{\pi}{2}$

4. Prepare the transformation matrix $[\boldsymbol{T}]$ in which, $\boldsymbol{a_{ij} = -sin\,\theta}$; $\boldsymbol{a_{ji} = sin\,\theta}$; $\boldsymbol{a_{ii} = cos\,\theta}$; $\boldsymbol{a_{jj} = cos\,\theta}$ remaining diagonal element is 1 and rest of the elements are 0.
5. Find transpose of the transformation matrix $[\boldsymbol{T}]^{\boldsymbol{T}}$.
6. Find new matrix given by $[\boldsymbol{A_1}] = [\boldsymbol{T}]^{\boldsymbol{T}}[\boldsymbol{A_0}]\,[\boldsymbol{T}]$.
7. Repeat the procedure from step 1 to 5 unless all off diagonal elements are eliminated.

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Diagonalization of symmetric matrix by Jacoby's Method

Example: $[A_0] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{21} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

1. *Select the largest (in absolute value) off diagonal element of matrix* $[A_0]$ *which is to be eliminated (say* $a_{ij} = a_{23} = |-1| = 1$, *which we want to eliminate).*
2. *Find the value of rotation angle* θ *using the expression,*

We have, $a_{23} = 1$; $a_{22} = 2$; *and* $a_{33} = 2$

$$tan\theta = \frac{\pm 2\, a_{ij}}{|a_{ii} - a_{jj}| + \sqrt{(a_{ii} - a_{jj})^2 + 4\, a_{ij}}} = \frac{\pm 2\,(1)}{|2-2| + \sqrt{(2-2)^2 + 4\,(1)}} = 1$$

Hence, $\theta = \frac{\pi}{4} = 45^o$; $\sin\theta = \sin\frac{\pi}{4} = 0.7071$ *and* $\cos\theta = \cos\frac{\pi}{4} = 0.7071$

Plus sign is used when $a_{ii} \geq a_{jj}$ *and minus sign is used when* $a_{ii} \leq a_{jj}$. *Also* $\frac{\pi}{2} \geq \theta \geq \frac{\pi}{2}$

1. *Prepare the transformation matrix* $[T]$ *in which,*

$a_{23} = -\sin\theta = -0.7071$; $a_{32} = \sin\theta = 0.7071$;

$a_{22} = \cos\theta = 0.7071$; $a_{33} = \cos\theta = 0.7071$

and remaining diagonal element is 1 and rest of the elements are 0.

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$$[T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\theta & -sin\theta \\ 0 & sin\theta & cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.7071 & -0.7071 \\ 0 & 0.7071 & 0.7071 \end{bmatrix}$$

Find transpose of the transformation matrix $[\mathbf{T}]^{\mathbf{T}}$

$$[T]^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.7071 & 0.7071 \\ 0 & -0.7071 & 0.7071 \end{bmatrix}$$

Find new matrix given by $[\mathbf{A_1}] = [\mathbf{T}]^{\mathbf{T}}[\mathbf{A_0}]\,[\mathbf{T}]$

$$[A_1] = [T]^T[A_0]\,[T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.7071 & 0.7071 \\ 0 & -0.7071 & 0.7071 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.7071 & -0.7071 \\ 0 & 0.7071 & 0.7071 \end{bmatrix}$$

$$[A_1] = \begin{bmatrix} 2 & -0.7071 & 0.7071 \\ -0.7071 & 0.999 & 0 \\ 0.7071 & 0 & 2.999 \end{bmatrix}$$

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$$[A_1] = \begin{bmatrix} 2 & -0.7071 & 0.7071 \\ -0.7071 & 0.999 & 0 \\ 0.7071 & 0 & 2.999 \end{bmatrix}$$

Select the largest (in absolute value) off diagonal element of matrix $[A_0]$ *which is to be eliminated*

(say $a_{ij} = a_{31} = 0.7071$*, which we want to eliminate).*

Find the value of rotation angle θ *using the expression,*

We have, $a_{33} = 3$ *;* $a_{22} = 2$ *; and* $a_{31} = 0.7071$

$$\tan\theta = \frac{\pm 2\, a_{ij}}{|a_{ii} - a_{jj}| + \sqrt{(a_{ii} - a_{jj})^2 + 4\, a_{ij}}} = \frac{\pm 2\,(0.7071)}{|3-2| + \sqrt{(3-2)^2 + 4\,(0.7071)}} = 0.478$$

Hence, $\theta = 25^o\, 33'$ *;* $\sin\theta = \sin 25^o\, 33' = 0.4312$ *and* $\cos\theta = \cos 25^o\, 33' = 0.9022$

$$[A_2] = [T]^T [A_1]\,[T] = \begin{bmatrix} 0.902 & 0 & -0.431 \\ 0 & 1 & 0 \\ 0.431 & 0 & 0.902 \end{bmatrix} \begin{bmatrix} 2 & -0.707 & 0.707 \\ -0.707 & 0.999 & 0 \\ 0.707 & 0 & 2.999 \end{bmatrix} \begin{bmatrix} 0.902 & 0 & 0.431 \\ 0 & 1 & 0 \\ -0.431 & 0 & 0.902 \end{bmatrix}$$

$$[A_2] = \begin{bmatrix} 1.63 & -0.63 & 0 \\ -0.63 & 1 & -0.30 \\ 0 & -0.30 & 3.36 \end{bmatrix}$$

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Diagonalization of symmetric matrix by Jacoby's Method

$$[A_2] = \begin{bmatrix} 1.63 & -0.63 & 0 \\ -0.63 & 1 & -0.30 \\ 0 & -0.30 & 3.36 \end{bmatrix}$$

Select the largest (in absolute value) off diagonal element of matrix $[A_0]$ *which is to be eliminated*

(say $a_{ij} = a_{21} = 0.63$*, which we want to eliminate).*

Find the value of rotation angle θ *using the expression,*

We have, $a_{22} = 1$; $a_{11} = 1.63$; *and* $a_{21} = 0.63$

$$tan\theta = \frac{\pm 2\, a_{ij}}{|a_{ii} - a_{jj}| + \sqrt{(a_{ii} - a_{jj})^2 + 4\, a_{ij}}} = \frac{\pm 2\,(0.63)}{|0.63| + \sqrt{(0.63)^2 + 4\,(0.63)}} = 0.5384$$

Hence, $\theta = 28^o\ 17'$; $sin\,\theta = sin\,28^o\ 17' = 0.4738$ *and* $cos\,\theta = cos\,28^o\ 17' = 0.8806$

$$[A_3] = [T]^T[A_2]\,[T] = \begin{bmatrix} 0.880 & -0.473 & 0 \\ 0.473 & 0.880 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1.63 & -0.63 & 0 \\ -0.63 & 1 & -0.30 \\ 0 & -0.30 & 3.36 \end{bmatrix} \begin{bmatrix} 0.880 & 0.473 & 0 \\ -0.473 & 0.880 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[A_3] = \begin{bmatrix} 2 & 0 & 0.14 \\ 0 & 0.61 & -0.26 \\ 0.14 & -0.26 & 3.36 \end{bmatrix}$$

Diagonalization of symmetric matrix by Jacoby's Method

$$[A_3] = \begin{bmatrix} 2 & 0 & 0.14 \\ 0 & 0.61 & -0.26 \\ 0.14 & -0.26 & 3.36 \end{bmatrix}$$

Select the largest (in absolute value) off diagonal element of matrix $[A_0]$ *which is to be eliminated (say* $a_{ij} = a_{32} = 0.26$*, which we want to eliminate).*

Find the value of rotation angle θ *using the expression,*

We have, $a_{33} = 3.36$; $a_{22} = 0.61$; *and* $a_{32} = 0.26$

$$tan\theta = \frac{\pm 2\, a_{ij}}{|a_{ii} - a_{jj}| + \sqrt{(a_{ii} - a_{jj})^2 + 4\, a_{ij}}} = \frac{\pm 2\,(0.26)}{|2.75| + \sqrt{(2.75)^2 + 4\,(0.26)}} = 0.0915$$

Hence, $\theta = 5^o\ 13'$; $sin\,\theta = sin\,5^o\ 13' = 0.090$ *and* $cos\,\theta = cos\,5^o\ 13' = 0.995$

$$[A_4] = [T]^T [A_3]\,[T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.995 & 0.090 \\ 0 & -0.090 & 0.995 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0.14 \\ 0 & 0.61 & -0.26 \\ 0.14 & -0.26 & 3.36 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.995 & -0.090 \\ 0 & 0.090 & 0.995 \end{bmatrix}$$

$$[A_4] = \begin{bmatrix} 2 & 0 & 0.138 \\ 0 & 0.578 & 0 \\ 0.138 & 0 & 3.34 \end{bmatrix}$$

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$$[A_4] = \begin{bmatrix} 2 & 0 & 0.138 \\ 0 & 0.578 & 0 \\ 0.138 & 0 & 3.34 \end{bmatrix}$$

Select the largest (in absolute value) off diagonal element of matrix $[A_0]$ *which is to be eliminated*

(say $a_{ij} = a_{31} = 0.138$*, which we want to eliminate).*

Find the value of rotation angle θ *using the expression,*

We have, $a_{33} = 3.34$ *;* $a_{11} = 2$ *; and* $a_{31} = 0.138$

$$tan\theta = \frac{\pm 2\, a_{ij}}{|a_{ii} - a_{jj}| + \sqrt{(a_{ii} - a_{jj})^2 + 4\, a_{ij}}} = \frac{\pm 2\,(0.138)}{|1.34| + \sqrt{(1.34)^2 + 4\,(0.138)}} = 0.096$$

Hence, $\theta = 5^o\, 29'$ *;* $sin\,\theta = sin\, 5^o\, 29' = 0.095$ *and* $cos\,\theta = cos\, 5^o\, 29' = 0.992$

$$[A_5] = [T]^T [A_4]\,[T] = \begin{bmatrix} 0.992 & 0 & -0.095 \\ 0 & 1 & 0 \\ 0.095 & 0 & 0.992 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0.138 \\ 0 & 0.578 & 0 \\ 0.138 & 0 & 3.34 \end{bmatrix} \begin{bmatrix} 0.992 & 0 & 0.095 \\ 0 & 1 & 0 \\ -0.095 & 0 & 0.992 \end{bmatrix}$$

$$[A_5] = \begin{bmatrix} 1.96 & 0 & 0 \\ 0 & 0.57 & 0 \\ 0 & 0 & 3.317 \end{bmatrix}$$

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Some Examples

$$[A_0] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1.96 & 0 & 0 \\ 0 & 0.57 & 0 \\ 0 & 0 & 3.317 \end{bmatrix}$$

$$[A_0] = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 3.00 & 0 & 0 \\ 0 & 1.585 & 0 \\ 0 & 0 & 4.414 \end{bmatrix}$$

$$[A_0] = \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

$$[A_0] = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

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$$[A_0] = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$[A_0] = \begin{bmatrix} 3 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$[A_0] = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$[A_0] = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$